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A Family of Dynamic Subgrid-Scale Models Consistent with Asymptotic Material Frame Indifference
「漸近MFIに整合するサブグリッドスケールモデル」

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§1. Introduction

In the large-eddy simulation[1] (LES) of incompressible turbulent flows, we require a subgrid-scale (SGS) model for the SGS stress tensor τ_{ij}, which is classically decomposed into three parts, as

$$\tau_{ij} \equiv \overline{u_i u_j} - \overline{u}_i \overline{u}_j = L_{ij} + C_{ij} + R_{ij}. \tag{1}$$

In the above expression, u_i is the i-th component of the velocity vector, and the overbar $\overline{f}$ denotes the grid-scale (GS) component of f, which is resolved by a spatial filtering operation defined using the filter function $G(\mathbf{x})$ as

$$\overline{f}(\mathbf{x}) = \int d\mathbf{x}' G(\mathbf{x} - \mathbf{x}') f(\mathbf{x}'). \tag{2}$$

Here, the three component parts, or the Leonard term L_{ij}, the cross term C_{ij}, and the SGS Reynolds stress R_{ij}, are respectively defined by

$$L_{ij} \equiv \overline{\overline{u}_i \overline{u}_j} - \overline{u}_i \overline{u}_j, \tag{3}$$

$$C_{ij} \equiv \overline{\overline{u}_i u'_j + u'_i \overline{u}_j}, \tag{4}$$

$$R_{ij} \equiv \overline{u'_i u'_j}, \tag{5}$$

where u'_i denotes the SGS part of u_i, defined by

$$u'_i \equiv u_i - \overline{u}_i. \tag{6}$$

The Leonard term L_{ij}, composed of the GS velocities, is resolvable; thus we require the SGS models for the cross term C_{ij}, and for the SGS Reynolds stress R_{ij}. The cross term is inherent in SGS modeling, while the SGS Reynolds stress is analogous to the Reynolds stress in the Reynolds stress closures.[2] Therefore, the neglect of the cross term with the direct evaluation of the Leonard term appears to be a way of modeling, and has in fact been performed in some large-eddy simulations.[3,4]

However, Speziale[5] pointed out that this approximation violates the Galilean invariance of the Navier-Stokes equation, though the neglect of the sum of the Leonard and the cross terms,

$$L_{ij} + C_{ij} \simeq 0, \tag{7}$$

is possible from the viewpoint of Galilean invariance. In this context, Germano[6] proposed a new decomposition of τ_{ij}, where each component term is Galilean invariant. It is composed of three terms again, the modified Leonard term L_{ij}^M, the modified cross term C_{ij}^M, and the modified SGS Reynolds stress R_{ij}^M:

$$\tau_{ij} = L_{ij}^M + C_{ij}^M + R_{ij}^M, \tag{8}$$

$$L_{ij}^M \equiv \overline{\overline{u}_i\overline{u}_j} - \overline{\overline{u}}_i\overline{\overline{u}}_j, \tag{9}$$

$$C_{ij}^M \equiv \overline{\overline{u}_i u'_j + u'_i\overline{u}_j} - \overline{\overline{u}}_i\overline{u'}_j - \overline{u'}_i\overline{\overline{u}}_j, \tag{10}$$

$$R_{ij}^M \equiv \overline{u'_i u'_j} - \overline{u'}_i\overline{u'}_j. \tag{11}$$

Only from the viewpoint of Galilean invariance can the modified cross term be neglected. In fact, the Bardina-type model[7] for the modified cross term should vanish by this constraint.[8]

However, it is put forth in the present paper that the neglect of the modified cross term is also inconsistent with the constraint of material frame indifference (MFI) in the limit of two-dimensional turbulence, as pointed out by Speziale.[9,10] In the following, this constraint is referred to as the asymptotic material frame indifference (AMFI). In the model expression for the Reynolds stress,[2] the lack of a term proportional to $(\partial u_i/\partial x_a - \partial u_a/\partial x_i)(\partial u_j/\partial x_a - \partial u_a/\partial x_j)$ is justified by the constraint of AMFI. Hereafter, the summation convention is used for repeated subscripts.

In the following, it is claimed in SGS modeling of incompressible turbulent flows that none of the (modified) Leonard terms, the (modified) cross terms, or their sums can be neglected, in principle, due to the constraint of AMFI, and that the model of Clark *et. al*[11] is consistent with this constraint. Furthermore, a family of dynamic SGS models consistent with this constraint is found, and specifically, a two-parameter dynamic SGS model is proposed as the most desirable member, whose expression for the SGS Reynolds stress asymptotically disappears in the limit of two-dimensional turbulence. These contents has been recently published in Shimomura.[12] In the present paper, the performances of the consistent dynamic SGS models in the real large-eddy simulations of rotating homogeneous turbulences are further reported.

In §**2** the the frame difference of the SGS stress tensor is reviewed, and in §**3** the impossibility of neglecting the (modified) Leonard terms, the (modified) cross terms, or their sums is proved based on the AMFI. In §**4** the consistent dynamic SGS models are proposed, and §**5** their superiority over the dynamic Smagorinsky model[4] is demonstrated in the large-eddy simulations of rotating homogeneous turbulences. Finally in §**6** the conclusions are summarized.

§2. Review of the frame difference of the SGS stress tensor

Here, let us review the findings of Speziale[13)] regarding the frame difference of the SGS stress tensor τ_{ij} under arbitrary time-dependent rotations of the reference frame specified by

$$x_i^* = Q_{ia}x_a, \tag{12}$$

where x_i is the position vector in an inertial frame, x_i^* is that in a rotating frame, and Q_{ij} is any time-dependent proper-orthogonal rotation matrix. Hereafter, as in (12), we denote the quantities in a rotating frame by adding the superscript $*$ to the notations of corresponding quantities in an inertial frame. From (12), we obtain the relation between the velocity components u_i in an inertial frame and the velocity components u_i^* in a rotating frame

$$Q_{ia}u_a = u_i^* + \epsilon_{iab}\Omega_a^* x_b^*, \tag{13}$$

where Ω_i^* is the angular velocity of the rotating frame, and ϵ_{ijk} is the alternating tensor. From (13) and the identity

$$\overline{x_i^*} = x_i^*, \tag{14}$$

we obtain

$$Q_{ia}\overline{u_a} = \overline{u_i^*} + \epsilon_{iab}\Omega_a^* x_b^*, \tag{15}$$

$$Q_{ia}u_a' = u_i^{*\prime}. \tag{16}$$

Accordingly, the modified Leonard term L_{ij}^M, the modified cross term C_{ij}^M, and the modified SGS Reynolds stress R_{ij}^M are respectively related to their counterparts as

$$Q_{ia}L_{ab}^M Q_{bj}^T = L_{ij}^{M*} + Z_{ij}^{L*}, \tag{17}$$

$$Q_{ia}C_{ab}^M Q_{bj}^T = C_{ij}^{M*} + Z_{ij}^{C*}, \tag{18}$$

$$Q_{ia}R_{ab}^M Q_{bj}^T = R_{ij}^{M*}, \tag{19}$$

where Q_{ij}^T denotes the transposed matrix of Q_{ij}. In (17) and (18), the terms Z_{ij}^{L*} and Z_{ij}^{C*} are given by

$$Z_{ij}^{L*} = \epsilon_{iab}\Omega_a^*(\overline{x_b^*\overline{\overline{u_j^*}}} - x_b^*\overline{\overline{u_j^*}}) + \epsilon_{jab}\Omega_a^*(\overline{\overline{\overline{u_i^*}}x_b^*} - \overline{\overline{u_i^*}}x_b^*) + \epsilon_{iab}\epsilon_{jcd}\Omega_a^*\Omega_c^*(\overline{x_b^*x_d^*} - x_b^*x_d^*), \tag{20}$$

$$Z_{ij}^{C*} = \epsilon_{iab}\Omega_a^*(\overline{x_b^*\overline{\overline{u_j^{*\prime}}}} - x_b^*\overline{\overline{u_j^{*\prime}}}) + \epsilon_{jab}\Omega_a^*(\overline{\overline{\overline{u_i^{*\prime}}}x_b^*} - \overline{\overline{u_i^{*\prime}}}x_b^*). \tag{21}$$

From (8) and (17)-(21), the SGS stress tensor τ_{ij} is written as

$$Q_{ia}\tau_{ab}Q_{bj}^T = \tau_{ij}^* + Z_{ij}^*, \tag{22}$$

$$Z_{ij}^* = Z_{ij}^{L*} + Z_{ij}^{C*} = \epsilon_{iab}\Omega_a^*(\overline{x_b^*u_j^*} - x_b^*\overline{u_j^*}) + \epsilon_{jab}\Omega_a^*(\overline{u_i^*x_b^*} - x_b^*\overline{u_i^*}) + \epsilon_{iab}\epsilon_{jcd}\Omega_a^*\Omega_c^*(\overline{x_b^*x_d^*} - x_b^*x_d^*). \tag{23}$$

Relations (17), (18), (22), and (19) state that the modified Leonard term L_{ij}^M, the modified cross term C_{ij}^M, and the modified SGS stress tensor τ_{ij} are frame different, but that the modified SGS Reynolds stress R_{ij}^M is frame indifferent.

Fureby and Tabor[14)] found, using the principle of frame indifference, that the filter function should possess spherical symmetry, i.e., $G = G(|\mathbf{x}|)$. The Gaussian filter with the filter width $\overline{\Delta}$, defined as

$$G(|\mathbf{x}|) = (\frac{\overline{\alpha}}{\pi})^{3/2} \exp(-\overline{\alpha} x_a x_a), \tag{24}$$

$$\overline{\alpha} \equiv \frac{6}{\overline{\Delta}^2}, \tag{25}$$

has this symmetry. For any filter function of the form $G = G(|\mathbf{x}|)$, the terms Z_{ij}^{L*}, Z_{ij}^{C*}, and Z_{ij}^* satisfy

$$\frac{\partial Z_{ia}^{L*}}{\partial x_a^*} = \frac{\partial Z_{ia}^{C*}}{\partial x_a^*} = \frac{\partial Z_{ia}^*}{\partial x_a^*} = 0, \tag{26}$$

since the solenoidal conditions hold:

$$\frac{\partial u_a^*}{\partial x_a^*} = \frac{\partial u_a^{*\prime}}{\partial x_a^*} = 0. \tag{27}$$

Essentially, these are the findings of Speziale.[13)]

Here we note from (17), (18), and (23) that the sum of L_{ij}^M and C_{ij}^M is frame different:

$$Q_{ia}(L_{ab}^M + C_{ab}^M)Q_{bj}^T = (L_{ij}^{M*} + C_{ij}^{M*}) + Z_{ij}^*. \tag{28}$$

Since the SGS Reynolds stress R_{ij} is frame indifferent as a result of (16),

$$Q_{ia}R_{ab}Q_{bj}^T = R_{ij}^*, \tag{29}$$

we find that the sum of L_{ij} and C_{ij} is also frame different, namely,

$$Q_{ia}(L_{ab} + C_{ab})Q_{bj}^T = (L_{ij}^* + C_{ij}^*) + Z_{ij}^*, \tag{30}$$

which is derived from (1), (22), and (29). By virtue of (26) and (30), we determine

$$Q_{ia}\frac{\partial}{\partial x_b}(L_{ab} + C_{ab}) = \frac{\partial}{\partial x_a^*}(L_{ia}^* + C_{ia}^*). \tag{31}$$

This describes the frame-indifferent feature of the term $\partial(L_{ia} + C_{ia})/\partial x_a$ that contributes to the filtered Navier-Stokes equation. Speziale[13)] required the SGS models to be compatible only with this feature and concluded that the neglect of $L_{ij} + C_{ij}$, or (7), is a possible way of modeling which is consistent with (31). However, this conclusion turns out to be false if we consider that the constraint of AMFI should be applied not only to the term $\partial(L_{ia} + C_{ia})/\partial x_a$ but also to the term $L_{ij} + C_{ij}$ itself, based on (30).

§3. Proof of the impossibility of neglecting $L_{ij}^{(M)}$, $C_{ij}^{(M)}$, or their sums

Now, we are ready to theoretically prove that none of the (modified) Leonard terms, the (modified) cross terms, or their sums can be neglected, in principle, due to the constraint of AMFI.

As is true of the Reynolds stress closures,[2] the model equation for the SGS stress tensor is asymptotically required to not depend on the angular velocity $\Omega^* = \sqrt{\Omega_a^* \Omega_a^*}$ of the reference frame in the limit of $\Omega^* \to \infty$ by the constraint of AMFI; this means that the dependence of velocity fields on Ω^* tends to disappear as Ω^* increases. Although the modified Leonard term L_{ij}^M is resolvable, the model equation of the modified cross term C_{ij}^M should not depend on Ω^*, neither should the modified SGS Reynolds stress R_{ij}^M, in the limit of $\Omega^* \to \infty$. If we denote the model for the modified cross term as $\Gamma_{ij}^M (\simeq C_{ij}^M)$, the corresponding model in a rotating frame can be derived from (18) as

$$C_{ij}^{M*} \simeq \Pi_{ij}^{M*} - Z_{ij}^{C*}, \tag{32}$$

where

$$\Pi_{ij}^{M*} = Q_{ia} \Gamma_{ab}^M Q_{bj}^T. \tag{33}$$

If the model is neglected ($\Gamma_{ij}^M = 0$), as in the Bardina-type model,[8] then $\Pi_{ij}^{M*} = 0$, from (33). Therefore, C_{ij}^{M*} does not obey this constraint, because model equation (32) is reduced to $C_{ij}^{M*} \simeq -Z_{ij}^{C*}$, which indicates the explicit dependence of C_{ij}^{M*} on Ω^*. It is impossible to neglect it. This logic, which proves the impossibility of neglecting the modified cross term, is not the same as, but is similar to, that of neglecting the cross term by the constraint of Galilean invariance, as pointed out by Speziale.[5] The term Z_{ij}^{C*} should be canceled out by a part of the term Π_{ij}^{M*}. In the same way, approximation (7) is found to be incompatible with the AMFI from (30).

There might be an objection to the AMFI because of the possibility that the Taylor-Proudman theorem does not hold in turbulent flows due to the survival of the non-negligible time-derivative of the velocity in the limit of $\Omega^* \to \infty$, which violates the geostrophic balance in the equation of motion. In this case, we have the following asymptotic equation instead of the geostrophic balance in the limit of $\Omega^* \to \infty$:

$$\frac{\partial \mathbf{u}^*}{\partial t} + 2\boldsymbol{\Omega}^* \times \mathbf{u}^* = -\nabla p^*,$$

where p is the pressure divided by the fluid density. If we choose the z-direction as the direction of $\boldsymbol{\Omega}$, then we have the plane-wave solution of the form

$$\mathbf{u}^* = \mathbf{u}_0^* \exp[i(\omega t - kx - ly - mz)],$$

$$\omega^2 = 4\Omega^{*2} m^2 (k^2 + l^2 + m^2)^{-1}.$$

This solution shows $|\mathbf{u}^*|$ remains finite even in the limit of $\Omega^* \to \infty$. Therefore, the above logic is expected to be valid in turbulent flows.

§4. Consistent SGS models

Most existing SGS models[4, 15-17] are not compatible with the constraint. Here, we derive a family of consistent SGS models for large-eddy simulations using the Gaussian filter.

It has been pointed out by Horiuti[18] that in the case of using Gaussian filter (24), the frame-different part Z_{ij}^* is analytically identical to

$$Z_{ij}^* = \frac{1}{2\overline{\alpha}}\left(\epsilon_{iab}\Omega_a^*\frac{\partial\overline{u_j^*}}{\partial x_b^*} + \epsilon_{jab}\Omega_a^*\frac{\partial\overline{u_i^*}}{\partial x_b^*} + \delta_{ij}\Omega_a^*\Omega_a^* - \Omega_i^*\Omega_j^*\right), \tag{34}$$

where δ_{ij} denotes the Kronecker delta. This identity is derived from the following formulae for the filtering operation with (24):

$$\overline{x_i^* x_j^*} = x_i^* x_j^* + \frac{1}{2\overline{\alpha}}\delta_{ij}, \tag{35}$$

$$\overline{x_i^* u_j^*} = x_i^*\overline{u_j} + \frac{1}{2\overline{\alpha}}\frac{\partial\overline{u_j^*}}{\partial x_i^*}. \tag{36}$$

As a SGS model that exactly satisfies the constraint of AMFI in the case of Gaussian filter (24), we turn to the model proposed by Clark *et al.*[11] (the Clark model) for the sum of the Leonard and the cross terms. It completely cancels the term Z_{ij}^* in (28) and is compatible with the AMFI. The model equation is given by

$$L_{ij} + C_{ij} \simeq \frac{1}{2\overline{\alpha}}\frac{\partial\overline{u_i}}{\partial x_a}\frac{\partial\overline{u_j}}{\partial x_a}. \tag{37}$$

Since (12) and (15) give

$$Q_{ia}\frac{\partial\overline{u_a}}{\partial x_b}Q_{bj}^T = \frac{\partial\overline{u_i^*}}{\partial x_j^*} + \epsilon_{iaj}\Omega_a^*, \tag{38}$$

we find that the Clark model (the right-hand side of (37)) has the same transformation property as that of the sum of the Leonard and cross terms in (30), or

$$Q_{ia}\frac{1}{2\overline{\alpha}}\frac{\partial\overline{u_a}}{\partial x_c}\frac{\partial\overline{u_b}}{\partial x_c}Q_{bj}^T = \frac{1}{2\overline{\alpha}}\frac{\partial\overline{u_i^*}}{\partial x_a^*}\frac{\partial\overline{u_j^*}}{\partial x_a^*} + Z_{ij}^*. \tag{39}$$

As a result of (30) and (39), Clark model (37) is form invariant under arbitrary time-dependent rotations of the reference frame, as well as under the extended Galilean group transformation.[5] Speziale[13] pointed out that the divergence of (37) is form invariant, but in the present paper, we find that the Clark model itself is form invariant. We are the first to point out that the Clark model is form invariant and consistent with the AMFI.

Here, we should be watchful of the terminology: "frame indifference" and "form invariance" are different concepts. "Frame indifference" is a property of a quantity such as the tensor f_{ij}, for example, which is related to the transformed quantity f_{ij}^* by $Q_{ia}f_{ab}Q_{bj}^T = f_{ij}^*$, whereas the "form invariance" is a property of an equation whose expression in a rotating frame has the same form as in an inertial frame, such as the (Galilean) principle of relativity. Even if $f_{ij} = g_{ij}$ holds for the frame-indifferent tensor f_{ij}, the tensor g_{ij} is not always frame indifferent, since the concept of frame indifference is not in the relation between f^*_{ij} and g^*_{ij}, but in that between g_{ij} and g^*_{ij}. If g^*_{ij} is frame indifferent, then the equation is form invariant, and if not, it is form variant for the frame-indifferent tensor f_{ij}.

The form invariance of the Clark model can be understood by noting that Z^*_{ij} in (34) is $O(\overline{\Delta}^2)$, and that the Clark model is the leading-order ($O(\overline{\Delta}^2)$) approximation for $L_{ij} + C_{ij}$ of the same order, which is derived from a Taylor expansion of the velocity with respect to the centerpoint of the filtering domain.[8] In this sense, the Clark model can be interpreted as the model for the sum of the modified Leonard and the modified cross terms, because the order of the last term $\overline{u'_i u'_j}$ on the right-hand side of (11) is estimated to be $O(\overline{\Delta}^4)$ according to this Taylor expansion. The compatibility with the AMFI and the Galilean invariance suggests that it is easier to model the sum of the Leonard term L_{ij} and cross term C_{ij} than to only model the latter while having the former directly calculated.

Because the Clark model relates to the sum of the (modified) Leonard and (modified) cross terms, the linear combinations with a compatible model for the (modified) SGS Reynolds stress forms a family of consistent SGS models for the total SGS stress tensor τ_{ij}.

The classical model for R^M_{ij} is the eddy-viscosity-type model.[3, 19] It is given by

$$\left(R^M_{ij}\right)_\Sigma \equiv R^M_{ij} - \frac{1}{3}R^M_{aa}\delta_{ij} \simeq -2\left(C_S\overline{\Delta}\right)^2 |\overline{S}|\overline{S}_{ij}, \tag{40}$$

where $\overline{S}_{ij}$ and $|\overline{S}|$ are the GS rate of strain tensor and its magnitude, defined as

$$\overline{S}_{ij} = \frac{1}{2}\left(\frac{\partial \overline{u}_i}{\partial x_j} + \frac{\partial \overline{u}_j}{\partial x_i}\right), \quad |\overline{S}| = \sqrt{2\overline{S}_{ab}\overline{S}_{ab}}, \tag{41}$$

and C_S is the model parameter. Hereafter, the term $(f_{ij})_\Sigma$ denotes the traceless tensor $f_{ij} - 1/3 f_{aa}\delta_{ij}$. This eddy-viscosity-type model is compatible with the AMFI since the GS rate of strain tensor S_{ij} is frame indifferent. Both the Bardina[7] and the filtered Bardina[20] models are also compatible. We note that the SGS algebraic model (SGSASM)[21] is another compatible model for R^M_{ij}, whose contribution systematically disappears as $\Omega^* \to \infty$.

In the framework of a dynamic SGS model,[4] we can also easily make SGS models consistent in the same way. For example, Clark model (37) with the dynamic Smagorinsky model is the simplest choice. It can reproduce a weakly compressible temporal mixing layer better than the dynamic Smagorinsky model; this was found by Vreman *et al.*[22] However, we should note that the property of the Clark model is not necessary but enough to be consistent with the AMFI that requires the independence of the model expression from Ω^* in the asymptotic limit of $\Omega^* \to \infty$; the Clark model has no explicit dependence on any finite Ω^*. Therefore, it may be better, for universal applicability of the model, to allow one more degree of freedom by introducing a modeling parameter as the coefficient of the right-hand side of (37), taking advantage of its dynamic procedure to automatically tune the modeling parameters.

Finally, we propose a consistent dynamic SGS model for the sum of the modified Leonard and modified cross terms as

$$\left(L_{ij}^M + C_{ij}^M\right)_\Sigma \simeq C_{LC}\frac{1}{2\overline{\alpha}}\left(\frac{\partial \overline{u_i}}{\partial x_a}\frac{\partial \overline{u_j}}{\partial x_a}\right)_\Sigma, \tag{42}$$

where C_{LC} is a dynamically determined model parameter. The counterpart of this model in a rotating frame is derived from (28) and (39) as

$$\left(L_{ij}^{M*} + C_{ij}^{M*}\right)_\Sigma \simeq C_{LC}\frac{1}{2\overline{\alpha}}\left(\frac{\partial \overline{u_i^*}}{\partial x_a^*}\frac{\partial \overline{u_j^*}}{\partial x_a^*}\right)_\Sigma + (C_{LC}-1)\left(Z_{ij}^*\right)_\Sigma. \tag{43}$$

In order to be compatible with the constraint of AMFI, the last term on the right-hand side of (43) should be asymptotically independent of Ω^* in the limit of infinite Ω^*, since the term Z_{ij}^* explicitly involves Ω_i^*. This is guaranteed by Lilly's[23] least squares method in the dynamic procedure for optimizing the parameters, on the condition that we linearly combine (42) to model τ_{ij} with a form invariant model for R_{ij}^M, such as the dynamic Smagorinsky model,[4] the dynamic (filtered) Bardina model,[7,20] or their linear combination. Thus, we can construct a family of dynamic SGS models which are consistent with the AMFI.

For example, let us formulate a two-parameter dynamic SGS model by combining (42) with the dynamic Smagorinsky model[4] as the least complex model. It is given in a rotating frame with one more parameter, C_R, by

$$\left(\tau_{ij}^*\right)_\Sigma \simeq C_{LC}\frac{1}{2\overline{\alpha}}\left(\frac{\partial \overline{u_i^*}}{\partial x_a^*}\frac{\partial \overline{u_j^*}}{\partial x_a^*}\right)_\Sigma - 2C_R\overline{\Delta}^2|\overline{S^*}|\overline{S^*}_{ij} + (C_{LC}-1)\left(Z_{ij}^*\right)_\Sigma. \tag{44}$$

In an inertial frame, the last term on the right-hand side of (44) disappears for $\Omega^* = 0$. If we apply Lilly's[23] least squares method to (44), we obtain the formula for C_{LC} and C_R,

$$\begin{pmatrix} C_{LC} \\ C_R \end{pmatrix} = \frac{1}{D}\begin{pmatrix} <\mathbf{MN}>_{tr}<\mathbf{MK}>_{tr} - <\mathbf{M}^2>_{tr}<\mathbf{KN}>_{tr} \\ <\mathbf{N}^2>_{tr}<\mathbf{MK}>_{tr} - <\mathbf{MN}>_{tr}<\mathbf{KN}>_{tr} \end{pmatrix}, \tag{45}$$

where

$$D = <\mathbf{M}^2>_{tr}<\mathbf{N}^2>_{tr} - <\mathbf{MN}>_{tr}^2. \tag{46}$$

In the above, **M**, **N**, and **K** denote the matrices M_{ij}, N_{ij}, and K_{ij}, respectively, and $<\mathbf{A}>_{tr}$ indicates the average of the trace of matrix **A** in the homogeneous domain. If we denote the test-filtered component as $\widetilde{f}$, the double-filter width as $\widetilde{\overline{\Delta}}$, and the corresponding coefficient in (25) as $\widetilde{\overline{\alpha}}$, they are defined by

$$M_{ij} = 2\overline{\Delta}^2\widetilde{|\overline{S^*}|\overline{S^*}_{ij}} - 2\widetilde{\overline{\Delta}}^2|\widetilde{\overline{S^*}}|\widetilde{\overline{S^*}}_{ij}, \tag{47}$$

$$N_{ij} = \frac{1}{2\overline{\alpha}}\widetilde{\left(\frac{\partial \overline{u_i^*}}{\partial x_a^*}\frac{\partial \overline{u_j^*}}{\partial x_a^*}\right)_\Sigma} - \frac{1}{2\widetilde{\overline{\alpha}}}\left(\frac{\partial \widetilde{\overline{u_i^*}}}{\partial x_a^*}\frac{\partial \widetilde{\overline{u_j^*}}}{\partial x_a^*}\right)_\Sigma + \left(1-\frac{\overline{\alpha}}{\widetilde{\overline{\alpha}}}\right)\left(\widetilde{Z_{ij}^*}\right)_\Sigma, \tag{48}$$

$$K_{ij} = \widetilde{\overline{u_i^*}\,\overline{u_j^*}} - \widetilde{\overline{u_i^*}}\,\widetilde{\overline{u_j^*}} - \left(1-\frac{\overline{\alpha}}{\widetilde{\overline{\alpha}}}\right)\left(\widetilde{Z_{ij}^*}\right)_\Sigma. \tag{49}$$

Expressions (34), (48), and (49) show that $N_{ij} \to (1-\overline{\alpha}/\widetilde{\overline{\alpha}})(\widetilde{\overline{Z^*_{ij}}})_\Sigma$, and $K_{ij} \to -(1-\overline{\alpha}/\widetilde{\overline{\alpha}})(\widetilde{\overline{Z^*_{ij}}})_\Sigma$, as Ω^* tends to infinity. Therefore, formula (45) leads to $C_{LC} \to 1$ and $C_R \to 0$ as $\Omega^* \to \infty$. As a result, in the limit of $\Omega^* \to \infty$, the dependence on Ω^* of model equation (44) for $(\tau^*_{ij})_\Sigma$ asymptotically disappears, which is consistent with the AMFI. Also, the model expression for R^M_{ij} is consistent with the two-dimensional turbulence.

§5. Comparison between SGS models in the large-eddy simulations of rotating homogeneous turbulences

In this section, we compare the performances of dynamic SGS models in the large-eddy simulation of rotating homogeneous turbulences. The three models are investigated: the dynamic Smagorinsky model (DSMG), the Clark model (37) with the dynamic Smagorinsky model (DCL), and the two-parameter dynamic model (44) (DTP). As shown in the previous section, both the DCL and the DTP are consistent with the constraint of AMFI, but the DSMG is not. Here, we note that the model expression of the DSMG in a rotating system is given by

$$\left(\tau^*_{ij}\right)_\Sigma \simeq -2C_R\overline{\Delta}^2|\overline{S^*}|\overline{S^*}_{ij} - \left(Z^*_{ij}\right)_\Sigma . \tag{50}$$

The numerical scheme is basically based on the spectral scheme though the second-order finite difference scheme is used the model part. All the LES calculations are done with 21^3 Fourier modes. The time is advanced by the fourth-order Runge-Kutta method. The initial data at $t = 1.10$ is obtained by filtering the DNS data in an inertial frame, which is in a fully-developed turbulent state with the Reynolds number based on the Taylor microscale 43.2. The rotation is abruptly applied to this initial state with the rotation number $R_o = k\Omega/\epsilon = 34.0$ at $t = 1.10$, where k and ϵ are the turbulent energy and its dissipation rate, respectively.

Fig. 1 shows the decays of GS turbulent energy in three models. The solid line is the result of the DSMG, and the broken line is the result of the DCL and DTP. (The DCL and the DTP shows the almost same results, whose difference is not resolved in the scale shown.) It is found that the DSMG shows the unphysical oscillation of GS turbulent energy while both the DCL and the DTP show monotonous decay. This is the fatal defect of the DSMG that is not consistent with the constraint of AMFI.

§6. Conclusions

By the constraint of AMFI, we find that none of the (modified) Leonard terms, the (modified) cross terms, or their sums can be neglected in principle in the SGS modeling of incompressible turbulent flows and that the model of Clark*et. al*[11] is consistent with this constraint. Furthermore, a family of dynamic SGS models consistent with this constraint is found, and specifically, a two-parameter dynamic SGS model is proposed as the most desirable member, whose expression for the

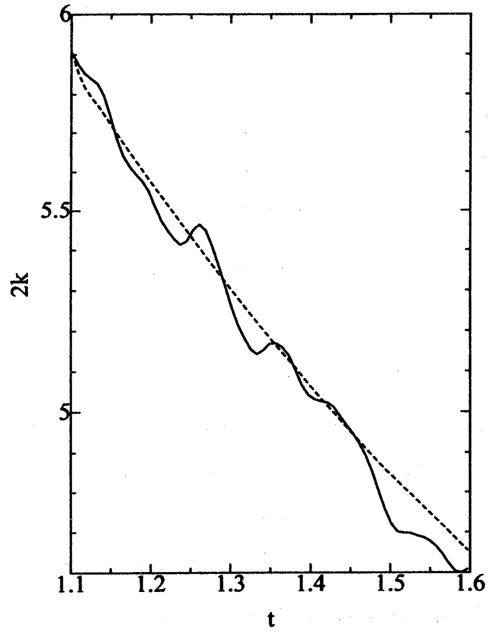

Fig. 1. The decay of GS turbulent energy at $\Omega = 50$: ———, DSMG; - - - - - -, DCL & DTP.

SGS Reynolds stress asymptotically disappears in the limit of two-dimensional turbulence. Their superiority over the dynamic Smagorinsky model[4] is demonstrated in the large eddy simulations of rotating homogeneous turbulences: the dynamic Smagorinsky model shows unphysical decay of the GS turbulent energy under an abrupt rotation, while the consistent models show a natural monotonous decay.

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